

Mathematics - Short Written Examination

Analysis, Sequences and Probability

Suggested time: 1 hour Calculator allowed

Instructions. Every answer must be justified. When a theorem is used, state it and verify its hypotheses. Final results must be clearly identified. Unless otherwise stated, give numerical approximations to three decimal places.

Indicative marking: Exercise 1: 7 marks; Exercise 2: 7 marks; Exercise 3: 6 marks.

Exercise 1 - Study of a function

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Consider the function f defined on $[0, 2]$ by

$$f(x) = e^x - x - 2.$$

1. Compute $f'(x)$ and study the variations of f on $[0, 2]$.
2. Prove that the equation $f(x) = 0$ has a unique solution α in $[0, 2]$. Then show that

$$1 < \alpha < 2.$$

3. Find an equation of the tangent to the graph of f at the point with x -coordinate 1.

Exercise 2 - A recursively defined sequence

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The sequence (u_n) is defined by

$$u_0 = 0 \quad \text{and} \quad u_{n+1} = \frac{u_n + 6}{4} \quad \text{for every } n \in \mathbb{N}.$$

1. Compute u_1 and u_2 .
2. Prove by induction that, for every $n \in \mathbb{N}$,

$$0 \leq u_n \leq 2.$$

3. Prove that the sequence (u_n) is increasing.
4. Deduce that (u_n) converges and determine its limit.

Exercise 3 - Quality control

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A company manufactures components using two machines, A and B .

- 65% of the components are produced by machine A ;
- 35% are produced by machine B ;
- among the components produced by A , 2% are defective;
- among the components produced by B , 5% are defective.

A component is selected at random. Let D be the event “the component is defective”.

1. Represent the situation using a probability tree.
2. Compute $P(D)$.
3. Given that the selected component is defective, compute the probability that it was produced by machine B .
4. Ten components are now selected independently under identical conditions. Let X be the number of defective components among the ten.
 - a. Justify that X follows a binomial distribution and state its parameters.
 - b. Compute the probability that at least one component is defective.

A clear solution allows the examiner to follow the reasoning without reconstructing it.