Written Mathematics Assessment

Exercice 1

Consider the function f defined on $[0;2]$ by $f(x) = e^x - x - 2$.

1) Compute $f'(x)$, then study the variations of f on $[0;2]$.

The function f is differentiable on $[0;2]$ as a sum of differentiable functions.

For every x in $[0;2]$, $f'(x) = e^x - 1$.

Now, on $[0;2]$, we have $e^x \geqslant 1$, with equality only for $x = 0$.

Hence $f'(x) \geqslant 0$ on $[0;2]$ and $f'(x) > 0$ on $]0;2]$.

x	0		2
$f'(x)$	0	$+$	
$f(x)$	-1	$\nearrow$	$e^2 - 4$

Thus, f is increasing on $[0;2]$.

2) Prove that the equation $f(x) = 0$ has a unique solution alpha in $[0;2]$. Then show that $1 < \text{alpha} < 2$.

The function f is continuous on $[0;2]$.

Moreover, $f(1) = e - 3 < 0$ and $f(2) = e^2 - 4 > 0$.

Since 0 lies between $f(1)$ and $f(2)$, the Intermediate Value Theorem ensures the existence of a real number alpha belonging to $]1;2[$ such that $f(\alpha) = 0$.

Since f is increasing on $[0;2]$, this solution is unique.

Therefore, the equation $f(x) = 0$ has a unique solution alpha on $[0;2]$, with $1 < \text{alpha} < 2$.

3) Find an equation of the tangent to the graph of f at the point with x-coordinate 1.

We have $f(1) = e - 3$ and $f'(1) = e - 1$.

An equation of the tangent at the point with x-coordinate 1 is therefore:

$$y = f'(1)(x-1) + f(1)$$
$$y = (e-1)(x-1) + e - 3$$
$$y = (e-1)x - 2$$

Thus, an equation of the tangent is $T : y = (e-1)x - 2$.

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Exercise 2

We define the sequence (u_n) by

$$u_0 = 0 \quad \text{and} \quad u_{n+1} = \frac{u_n + 6}{4} \quad \text{for every } n \text{ in } \mathbb{N}.$$

1) Compute u_1 and u_2.

We have :

$$u_1 = \frac{u_0 + 6}{4} = \frac{0 + 6}{4} = \frac{3}{2}.$$

Then

$$u_2 = \frac{u_1 + 6}{4} = \frac{\frac{3}{2} + 6}{4} = \frac{\frac{15}{2}}{4} = \frac{15}{8}$$

Hence $u_1 = \frac{3}{2}$ and $u_2 = \frac{15}{8}$.

2) Prove by induction that, for every n in $\mathbb{N}$, $0 \leqslant u_n \leqslant 2$.

Initial step : for $n = 0$, we have $u_0 = 0$, so $0 \leqslant u_0 \leqslant 2$. The property is true at rank 0.

Induction step : let n be in $\mathbb{N}$ and assume that $0 \leqslant u_n \leqslant 2$.

Then $6 \leqslant u_n + 6 \leqslant 8$.

Dividing by 4, which is positive, we obtain $\frac{3}{2} \leqslant u_{n+1} \leqslant 2$.

In particular, $0 \leqslant u_{n+1} \leqslant 2$.

Thus the property is hereditary.

Conclusion : By the principle of mathematical induction,

$$\text{for every } n \text{ in } \mathbb{N}, \quad 0 \leqslant u_n \leqslant 2.$$

3) Prove that the sequence (u_n) is increasing.

For every n in $\mathbb{N}$,

$$\begin{aligned} u_{n+1} - u_n &= \frac{u_n + 6}{4} - u_n \\ &= \frac{u_n + 6 - 4u_n}{4} \\ &= \frac{6 - 3u_n}{4} \\ &= \frac{3(2 - u_n)}{4} \end{aligned}$$

Now, by the previous question, $u_n \leqslant 2$.

Hence $2 - u_n \geqslant 0$, and therefore $u_{n+1} - u_n \geqslant 0$.

Thus the sequence (u_n) is increasing.

4) Deduce that (u_n) converges and determine its limit.

The sequence (u_n) is increasing and bounded above by 2.

By the theorem on monotone sequences, it converges. Let ℓ be its limit.

Passing to the limit in the recurrence relation, we obtain :

$$\ell = \frac{\ell + 6}{4}$$

Hence $4\ell = \ell + 6$, then $3\ell = 6$, so $\ell = 2$.

Thus, the sequence (u_n) converges to 2.

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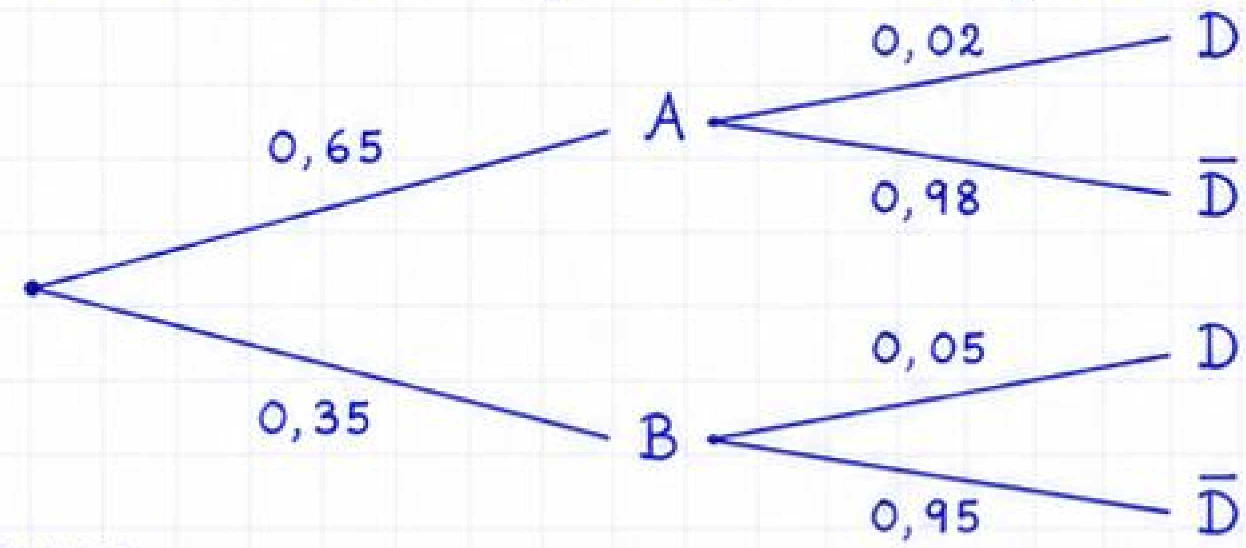
Exercice 3

A company manufactures components using two machines A and B.
We denote by D the event : "the component is defective".
We have : $P(A) = 0,65$; $P(B) = 0,35$; $P_A(D) = 0,02$; $P_B(D) = 0,05$.

1) Represent the situation using a probability tree.



2) Compute $P(D)$.
By the law of total probability,

$$\begin{aligned} P(D) &= P(A \cap D) + P(B \cap D) \\ &= P(A) \times P_A(D) + P(B) \times P_B(D) \\ &= 0,65 \times 0,02 + 0,35 \times 0,05 \\ &= 0,013 + 0,0175 \\ &= 0.0305 \end{aligned}$$

Hence $P(D) = 0.0305$.

3) Given that the selected component is defective, compute the probability that it was produced by machine B.
We are looking for $P_D(B)$.
By the formula for conditional probabilities,

$$P_D(B) = \frac{P(B \cap D)}{P(D)} = \frac{0,35 \times 0,05}{0,0305} = \frac{0,0175}{0,0305} = \frac{35}{61} \approx 0,574$$

Hence the required probability is approximately 0.574.

4) Ten components are now selected independently,
We denote by X the number of defective components among the 10.

a) Justify that X follows a binomial distribution and state its parameters.
Each draw is independent and the probability that a component is defective is $p = 0.0305$.
Therefore X follows the binomial distribution $B(10 ; 0,0305)$.

b) Compute the probability that at least one component is defective.
We are looking for $P(X \geq 1)$.
We use the complementary event :

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - (1 - 0,0305)^{10} \\ &= 1 - 0,9695^{10} \\ &\approx 0,266 \end{aligned}$$

Thus, the probability that at least one component is defective is approximately 0.266.